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THE STRUCTURAL PROPERTIES OF CEPHEID LIGHT CURVES

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ABSTRACT

Fourier decompositions are performed for the V magnitude variations of a sample of 57 Cepheids ranging in period from 2 to 17 days. The coefficients have been tabulated to allow rapid reconstruction of the light curves. It is shown that combinations of the low-order coefficients quantitatively describe the progression of curve shape with period. The pulsation amplitude plays only a minor role. Sharp breaks in the progression, occurring at around 10 days, are attributed to the resonance $P_2/P_0=0.5$. It is suggested that the Fourier decompositions provide a straightforward method for resolving a number of controversies concerning theoretical interpretations of the Hertzsprung sequence. The paper closes with a brief discussion of three short-period stars which seem to occupy a unique position in our sample.

Subject headings: stars: Cepheids — stars: pulsation

I. INTRODUCTION

It has long been known that the light curves of classical Cepheids display a considerable regularity of structure. The progression of curve shape with period, known as the Hertzsprung sequence, has been widely discussed in the literature (e.g., Ledoux and Walraven 1958). In recent years, with the advent of nonlinear pulsation codes, a number of attempts have been made to compare observed light curves with those produced by theoretical models. These investigations, beginning with the work of Christy (1966, 1968) and Stobie (1969*a*, *b*), have generally given rise to a "bump-mass" anomaly, wherein the masses indicated by the Hertzsprung sequence are almost a factor of 2 smaller than those emerging from the theory of stellar evolution. This problem has been reviewed in detail by Cox (1980).

Various possibilities exist for ameliorating the mass discrepancy, including: chemical inhomogeneities in the models (Cox, Michaud, and Hodson 1978); adoption of the Carson opacities (Vermury and Stothers 1978); and inclusion of magnetic fields (Stothers 1979). Up to the present time, none of these remedies has been widely accepted. Indeed, the question has been raised as to whether the bump-mass anomaly is really serious enough to warrant heroic measures (Cox 1978). The latter objection may be proposed on at least two grounds. In the first place, theoretical light curves (and to a certain extent velocity curves as well) often exhibit nonphysical features which are artifacts of the computation. This situation has recently been improved by the introduction of new numerical techniques (Davis and Davison 1978).

On the other hand, progress has also been hampered because the observations themselves have not been presented in a form which facilitates comparison with theory. While there is general agreement that the Hertzsprung sequence is real, its definition has generally remained qualitative. Thus the literature is replete with references to ill-defined qualities such as "bumps," "shoulders," and "standstills." Even in the oft-quoted study by Fricke, Stobie, and Strittmatter (1972), bump phases were determined graphically, presumably being read off from more or less crude curves drawn through the observed points. If possible, it would clearly be useful to improve these methods.

II. THE FOURIER DECOMPOSITIONS

In the present investigation, we attempt to get a quantitative handle on the shapes of Cepheid light curves, using the technique of Fourier decomposition. While similar studies have been made in the past, notably by Schaltenbrand and Tamann (1971), the Fourier coefficients themselves have generally not been published, nor has any attempt been made to describe detailed structural properties of the observed curves.

The Fourier fitting program employed in this work has been discussed by Simon (1979). In its single-period mode, fits to the observed magnitudes have the form

$$A_0+A_i\cos\left[i\omega(t-t_0)+\phi_i\right], \qquad (1)$$

where, for a given fit, the index i runs from 1 to $i_{\max}(2\leq i_{\max}\leq 8)$. Thus the program has the capability of using coefficients as high as eighth order. The time t in equation (1) is employed in the form

$$t = \mathrm{JD} - \tau,$$

where JD is the time of the observation in Julian days. The quantities τ and t_0 (eq. [1]) are constants. The former has a single value for all observations from the same source, while the latter is different for each observed star.

The list of stars chosen for study is given in Table 1, numbered in order of increasing period. A large majority of the sample comes either from the Tonantzintla catalog (Mitchell *et al.* 1964) or from the work of Pel (1976). These references, indicated as "T" and "P" respectively, are listed with other source references in the third column of Table 1. The rest of the columns in Table 1 show the period of the observed star, its amplitude, the number of observations employed by the fitting routine, the order of the best fit, and the standard deviation of this fit multiplied by 100. The periods are generally those given by the source reference, although in some cases it was found that a better fit could be obtained with a somewhat altered period. It should be emphasized here that because the quantities which interest us are essentially independent of small inaccuracies in period, no attempt was made to refine the periods with high precision. Thus the values published in Table 1 should not be considered definitive.

For most of the stars in our sample, a fourth-order fit was deemed sufficient and appropriate. The criteria employed to judge the various fits were subjective. Among them, three weighed most heavily: (1) the standard deviation should be reasonably small, particularly in comparison with the amplitude; (2) the low-order coefficients $A_i, \phi_i (i=1,2,3)$ should not change drastically as the order of the fit is increased; and (3) the fit should look appropriate to the eye, i.e., the Fourier curve ought to pass close to the observed points. Stars which failed badly one or more of the above standards were thrown out of the sample. If a set of observations could be fitted satisfactorily with both fourth-order and eighth-order Fourier series, the former was always selected unless the passage to higher order improved the fit substantially. In two cases, second-order fits were judged sufficient.

In Table 2 we list the Fourier coefficients up to the fourth order for most of the stars in our sample. The time parameter t_0 for each star is given in the third column. An *a* in the first column indicates that an eighth-order fit has been constructed. The additional Fourier coefficients for these stars are presented in Table 3. A *b* in the second column of Tables 2 and 3 appears as a warning that the corresponding Fourier decomposition, while accurate enough to determine the lower-order coefficients (see below), may not furnish a competely adequate description of all the nuances of the light curve. These cases can be improved with more observations. Omitted entirely from Tables 2 and 3 are those stars with observed points taken from more than one source.

Our quantification of the light curve shapes focuses on the lower-order Fourier coefficients. Let us define the quantities

$$R_{21} = A_2/A_1,$$

$$R_{31} = A_3/A_1,$$

$$\phi_{21} = \phi_2 - 2\phi_1,$$

$$\phi_{31} = \phi_3 - 3\phi_1.$$

The first two quantities measure the relative importance of the second and third Fourier terms respectively, while the latter two describe the phase differences between the second and third terms and the leading term. We note that ϕ_{21} and ϕ_{31} are defined so as to be independent of both time translations and the basic frequency ω.

Figure 1 shows a plot of ϕ_{21} versus period for our complete sample of 57 Cepheids. The stars from 3 to 8 days show a striking regularity, culminating in a sharp break between 9 and 11 days. Thus it is seen that, at the shorter periods, the quantity ϕ_{21} describes a readily discernible quantitative progression.

In our view, the near discontinuity centered at about 10 days is attributable to the resonance $P_2/P_0 = 0.5$ (Simon and Schmidt 1976). Figure 1 furnishes the most dramatic illustration to date of the resonant interaction. In this regard, it should be recalled that ϕ_{21} is exactly the quantity predicted to display resonance effects according to the iterative theory of Simon (1977).

For periods ≥ 11 days, the phase difference ϕ_{21} drops back to previous levels, but now displaying considerable scatter with no progression easily marked. We note in this connection that the stars between 3 and 8 days tended to have smooth data with good phase coverage and little noise. Furthermore, at the shorter periods the second Fourier term (frequency 2ω) was generally an important one. Both of these circumstances make for accurate determination of ϕ_{21}. On the other hand, for $P > 9$ days, the observations were usually poorer and the second Fourier term less important compared with higher terms.

The latter result is demonstrated in Figure 2 which displays a plot of the amplitude ratio R_{21} versus period. One notices immediately the large values of R_{21} attained at short periods, as well as a precipitous drop in the vicinity of 9–10 days and subsequent recovery as the periods get longer. Since R_{21} measures the relative strength of a higher and a lower Fourier term, we might predict a correlation with amplitude, such that R_{21} would be greater in the higher amplitude stars, which depart the most from linearity. However, this does not turn out to be the case. In a plot of R_{21} versus ampli-

TABLE 1

LIST OF STARS IN THE SURVEY

Star No.	Star Name	Source	Period	Amp. (Mag.)	No. of Obs.	Order of Fit	Standard Deviation (x 10^2)
1	SU Cas	1	1.94932	0.42	62	4	1.81
2	EU Tau	2	2.10250	0.35	239	2	1.02
3	UY Eri	P	2.213235	0.68	36	4	1.87
4	RT Mus	3	3.08617	0.76	33	4	1.86
5	VZ CMa	3	3.12608	0.42	38	2	1.28
6	BY Cas	T	3.22282	0.42	62	4	2.36
7	R Tra	P	3.389287	0.56	32	4	0.73
8	GU Nor	P	3.45324	0.59	34	4	1.54
9	SS Sct	P	3.671253	0.52	30	4	0.75
10	UX Car	P	3.682246	0.78	30	4	1.63
11	SY Cas	T	4.07159	0.86	64	4	2.23
12	V496 Cen	P	4.42413	0.59	29	4	0.96
13	XY Cas	T	4.49933	0.58	51	8	2.85
14	V482 Sco	P	4.52786	0.64	33	4	0.85
15	TV CMa	P	4.66970	0.77	29	4	1.61
16	RY CMa	4	4.67825	0.72	55	4	3.01
17	V381 Cen	T	5.07865	0.72	26	4	2.73
18	SW Cas	T	5.44100	0.74	70	4	3.35
19	FM Cas	T	5.80910	0.60	58	4	1.59
20	ST Vel	P	5.8584249	0.66	30	4	1.36
21	VV Cas	T	6.20741	0.91	60	4	2.90
22	BP Cas	T	6.27238	0.79	39	4	2.48
23	RS Cas	T	6.29561	0.79	77	4	2.04
24	RR Lac	T	6.41614	0.87	59	4	3.80
25	AY Sgr	P	6.56959	0.85	32	4	1.86
26	U Sgr	T	6.74531	0.83	148	8	4.46
27	V496 Aql	T	6.807486	0.41	41	4	2.51
28	AK Cep	T	7.23140	0.67	39	4	2.56
29	V600 Aql	P	7.23845	0.66	29	4	1.42
30	V336 Aql	P	7.303552	0.74	35	4	1.51
31	CK Sct	P	7.41522	0.51	36	4	1.03
32	W Sgr	T	7.59511	0.84	60	4	4.64
33	GH Cyg	T	7.81834	0.82	49	4	2.95
34	U Vul	T	7.99040	0.77	51	4	3.22
35	S Sge	T	8.38205	0.94	64	4	3.73
36	TX Mon	P	8.701731	0.66	34	4	1.77
37	PZ Aql	P	8.7513	0.79	43	4	2.36
38	GH Lup	P	9.2780	0.19	37	4	0.62
39	FN Aql	P	9.48224	0.60	44	8	1.13
40	CR Car	P	9.7617	0.64	36	8	1.90
41	DD Cas	T	9.81040	0.61	39	4	1.92
42	β Dor	T	9.84295	0.66	53	8	2.40
43	BZ Cyg	T	10.1408	0.55	46	4	2.95
44	ζ Gem	5	10.15374	0.49	46	4	2.28
45	Z Lac	T	10.8854	1.02	66	4	5.05
46	VX Per	T	10.911	0.75	41	4	3.93
47	SV Per	T	11.130	0.80	34	4	2.85
48	DR Vel	P	11.20000	0.71	40	4	1.64
49	RY Cas	T	12.1339	1.04	61	8	2.49
50	U Nor	P	12.64133	1.02	34	8	1.33
51	TT Aql	P	13.7546	1.12	47	8	1.63
52	FI Car	P	13.4542	0.79	37	4	3.43
53	TX Cyg	T	14.7178	1.26	39	8	3.09
54	RW Cas	T	14.7954	1.27	58	8	5.41
55	SZ Cyg	T	15.1093	0.91	49	8	2.71
56	RW Cam	T	16.4144	0.86	44	8	2.32
57	Y Oph	T	17.12210	0.59	76	4	2.89

T) Mitchell, et al. (1964). τ = 2430000
P) Pel (1976). Prior to fitting, data points were converted to UBV system using Pel's formula (p. 419). τ = 2440000.
1) Mitchell, et al. (1964); Niva and Schmidt (1979).
2) Guinan (1972); Sanwal and Parthasarathy (1974).
3) Stobie and Balona (1979). τ = 2430000.
4) Dean, et al. (1977). τ = 2440000.
5) Mitchell, et al. (1964). Internal scale readjustments made before fit. Star does not appear in Table 2.

TABLE 2

Fourier Coefficients (A_i, ϕ_i) (where $i = 1-4$)

Star No.	Star Name	t_0	A_0	A_1	ϕ_1	A_2	ϕ_2	A_3	ϕ_3	A_4	ϕ_4
3	UY Eri	950	11.34	2.72(-1)	4.56	1.12(-1)	1.05	4.46(-2)	4.00	1.87(-2)	5.13(-1)
4	RT Mus	175	9.03	3.31(-1)	2.77(-2)	1.31(-1)	4.13	5.03(-2)	2.24	4.16(-2)	6.16
5	VZ CMa	175	9.39	2.07(-1)	9.10(-1)	1.11(-2)	5.83	--	--	--	--
6	BY Cas	6800	10.36	1.84(-1)	6.94(-1)	1.40(-1)	5.38	6.85(-3)	4.15(-1)	4.39(-3)	4.43(-1)
7	R Tra	5000	6.67	2.51(-1)	5.43	7.06(-2)	2.72	2.58(-2)	3.46(-1)	2.38(-2)	4.09
8	GU Nor[b]	850	10.43	2.44(-1)	4.21	8.02(-2)	5.12(-2)	3.80(-2)	2.06	1.68(-2)	3.94
9	SS Sct	800	8.24	2.33(-1)	1.75	6.89(-2)	1.38	2.58(-2)	1.16	8.23(-3)	7.88(-1)
10	UX Car	750	8.35	3.52(-1)	1.71	1.27(-1)	1.30	5.91(-2)	9.59(-1)	2.85(-2)	7.48(-1)
11	SY Cas	6800	9.90	3.40(-1)	9.54(-1)	1.24(-1)	6.21	6.01(-2)	5.11	3.12(-2)	4.20
12	V496 Cen[b]	850	9.99	2.71(-1)	4.58	8.38(-2)	8.89(-1)	2.96(-2)	3.70	8.43(-3)	3.39(-1)
13[a]	XY Cas	6800	9.98	2.51(-1)	4.75	8.74(-2)	1.30	2.08(-2)	4.08	1.12(-2)	1.21(-1)
14	V482 Sco	750	8.00	2.81(-1)	2.62	9.44(-2)	3.25	3.07(-2)	3.99	1.72(-2)	4.54
15	TV CMa	800	10.62	3.27(-1)	5.46	1.28(-1)	2.63	5.95(-2)	6.14	1.95(-2)	3.24
16	RY CMa	2000	8.13	3.02(-1)	4.74	1.09(-1)	1.28	4.51(-2)	4.14	1.72(-2)	2.35(-1)
17	V381 Cen	5000	7.69	3.00(-1)	4.70	9.91(-2)	1.28	3.64(-2)	4.07	1.42(-2)	1.01
18	SW Cas	1200	9.71	2.87(-1)	3.63	1.01(-1)	5.35	3.20(-2)	7.62(-1)	1.19(-2)	4.96
19	FM Cas	6800	9.13	2.41(-1)	3.28	9.15(-2)	4.90	2.23(-2)	2.74(-2)	1.13(-2)	2.53
20	ST Vel	800	9.73	2.82(-1)	5.56	1.11(-1)	3.16	2.99(-2)	4.16(-1)	1.08(-2)	4.14
21	VV Cas	6800	10.78	3.61(-1)	4.98	1.49(-1)	1.94	5.76(-2)	4.85	3.40(-2)	1.95
22	BP Cas	6900	10.95	3.26(-1)	5.10	1.23(-1)	2.29	4.09(-2)	5.30	2.29(-2)	3.36
23	RS Cas	6800	9.97	3.52(-1)	3.81	1.30(-1)	5.93	4.35(-2)	1.41	2.07(-2)	3.76
24	RR Lac	6800	8.88	3.49(-1)	5.69	1.24(-1)	3.35	3.06(-2)	6.75(-1)	1.78(-2)	4.12
25	AY Sgr	800	10.59	3.65(-1)	1.03	1.40(-1)	4.92(-1)	4.49(-2)	5.68	1.72(-2)	5.10
26[a]	U Sgr[b]	6000	6.73	3.05(-1)	2.60	1.07(-1)	3.65	4.52(-2)	4.03	5.83(-3)	2.38
27	V496 Aql	6000	7.78	1.69(-1)	5.69	3.85(-2)	3.54	9.96(-3)	2.01	7.66(-3)	1.86
28	AK Cep	6800	11.21	2.85(-1)	3.86	1.06(-1)	6.15	3.60(-2)	1.83	1.55(-2)	4.44
29	V600 Aql	900	10.08	2.75(-1)	3.31	9.50(-2)	5.30	3.45(-2)	1.76(-1)	1.67(-2)	5.89
30	V336 Aql[b]	900	9.88	3.15(-1)	1.67	1.01(-1)	1.99	5.39(-2)	1.56	1.36(-2)	4.92
31	CK Sct	1050	10.61	2.11(-1)	4.10	5.63(-2)	7.01(-1)	1.99(-2)	2.59	9.62(-3)	3.10
32	W Sgr[b]	5000	4.70	3.38(-1)	4.42	1.04(-1)	1.36	7.35(-2)	3.78	2.22(-2)	4.93
33	GH Cyg	6800	9.94	3.36(-1)	6.65(-1)	9.14(-2)	2.34(-2)	4.95(-2)	4.96	2.70(-2)	2.60
34	U Vul[b]	6000	7.16	3.20(-1)	4.08	6.55(-2)	4.08(-1)	5.08(-2)	2.52	1.96(-2)	3.60
35	S Sge	5000	5.66	3.45(-1)	1.73	8.25(-2)	2.24	6.06(-2)	2.27	4.51(-2)	7.87(-1)
36	TX Mon	850	10.97	2.91(-1)	2.55	5.70(-2)	4.30	4.48(-2)	5.79	2.54(-2)	5.43
37	PZ Aql	900	11.70	3.65(-1)	4.66	4.59(-2)	1.75	2.08(-2)	2.76	1.88(-2)	5.98
38	GH Lup	1100	7.64	9.15(-2)	4.85	5.86(-3)	3.76	1.53(-3)	4.53	3.40(-3)	4.85
39[a]	FN Aql[b]	950	8.41	2.69(-1)	4.25	5.91(-3)	3.49	2.14(-2)	6.16(-1)	1.43(-2)	3.54
40[a]	CR Car	850	11.58	2.53(-1)	3.96	3.77(-2)	5.07(-1)	2.09(-2)	4.50	1.91(-2)	4.19
41	DD Cas	6800	9.87	2.63(-1)	6.11	1.48(-2)	5.07(-1)	2.53(-2)	5.71	1.19(-2)	4.06
42[a]	β Dor[b]	5000	3.77	2.77(-1)	3.16	1.80(-2)	1.15(-2)	2.22(-2)	2.92	1.69(-2)	3.45
43	BZ Cyg[b]	6800	10.24	2.41(-1)	5.56	4.76(-2)	3.96	2.59(-2)	2.35	1.43(-2)	4.50
45	Z Lac[b]	6000	8.46	3.88(-1)	1.77	4.13(-2)	1.28	4.41(-2)	5.75	4.40(-2)	5.37
46	VX Per[b]	6800	9.32	3.13(-1)	2.68	3.58(-2)	4.84	2.43(-2)	6.76(-1)	2.04(-2)	6.26
47	SV Per	6000	8.98	3.30(-1)	2.46(-1)	2.72(-2)	5.08	3.67(-2)	1.40	3.57(-2)	6.08
48	DR Vel	1000	9.55	3.21(-1)	2.54	1.73(-2)	2.00	3.33(-2)	1.63	2.51(-2)	2.68
49[a]	RY Cas	6800	9.99	3.93(-1)	3.69	7.65(-2)	5.15	4.75(-2)	5.59	4.18(-2)	2.50(-1)
50[a]	U Nor[b]	750	9.30	4.06(-1)	2.67(-1)	7.22(-2)	4.74	4.76(-2)	1.82	4.50(-2)	5.54
51[a]	TT Aql	950	7.20	4.35(-1)	2.37	9.80(-2)	2.96	5.83(-2)	2.39	6.23(-2)	2.57
52	FI Car[b]	1050	11.63	3.26(-1)	3.45(-1)	2.96(-2)	4.76	5.14(-2)	1.44	2.20(-2)	6.08
53[a]	TX Cyg	6800	9.56	4.63(-1)	5.31	1.18(-1)	2.40	8.43(-2)	4.96	6.95(-2)	1.93
54[a]	RW Cas	5000	9.27	4.60(-1)	3.09	1.07(-1)	4.57	5.54(-2)	4.71	6.35(-2)	5.92
55[a]	SZ Cyg	6800	9.47	3.79(-1)	4.85	8.36(-2)	1.35	5.72(-2)	3.34	5.34(-2)	5.94
56[a]	RW Cam[b]	6850	8.67	3.48(-1)	3.23	9.69(-2)	4.81	4.94(-2)	4.67	5.80(-2)	3.48(-1)
57	Y Oph	6000	6.21	2.28(-1)	2.62	4.39(-2)	3.30	2.46(-2)	3.40	1.56(-2)	3.53

[a]Eighth-order fit. See Table 3.

[b]Fourier decomposition may not fully describe light curve.

TABLE 3

FOURIER COEFFICIENTS (A_i, ϕ_i) (where $i = 5-8$)

Star No.	Star Name	A_5	ϕ_5	A_6	ϕ_6	A_7	ϕ_7	A_8	ϕ_8
13	XY Cas	7.69(-3)	4.28	4.07(-3)	3.85	6.77(-3)	8.17(-2)	4.93(-3)	5.88
26	U Sgr[b]	8.44(-3)	6.26	1.42(-2)	3.76	1.04(-2)	4.22	5.59(-3)	5.41
39	FN Aql[b]	3.03(-3)	5.00	8.11(-3)	5.65	6.74(-3)	1.28	9.64(-3)	3.26
40	CR Car	2.14(-2)	8.88(-1)	1.26(-2)	6.08	1.24(-2)	2.70	8.60(-3)	1.13
42	β Dor[b]	1.56(-2)	3.44	1.28(-2)	4.12	7.60(-3)	4.63	8.14(-3)	1.91(-1)
49	RY Cas	2.53(-2)	1.37	1.40(-2)	2.30	9.15(-3)	2.81	3.11(-3)	4.02
50	U Nor[b]	3.61(-2)	3.20	2.02(-2)	9.62(-1)	6.79(-3)	4.80	6.74(-2)	2.86
51	TT Aql	4.21(-2)	2.99	2.65(-2)	3.54	2.49(-2)	4.01	1.37(-2)	4.02
53	TX Cyg	6.05(-2)	5.14	5.12(-2)	1.92	2.94(-2)	4.54	2.28(-2)	1.47
54	RW Cas	4.83(-2)	6.88(-1)	5.39(-2)	1.80	4.07(-2)	3.19	2.30(-2)	4.24
55	SZ Cyg	2.62(-2)	2.53	2.30(-2)	5.83	2.50(-2)	3.19	7.37(-3)	5.48
56	RW Cam[b]	3.26(-2)	1.20	4.12(-2)	3.18	1.54(-2)	3.22(-2)	1.86(-2)	5.47

[b]Fourier decomposition may not fully describe light curve.

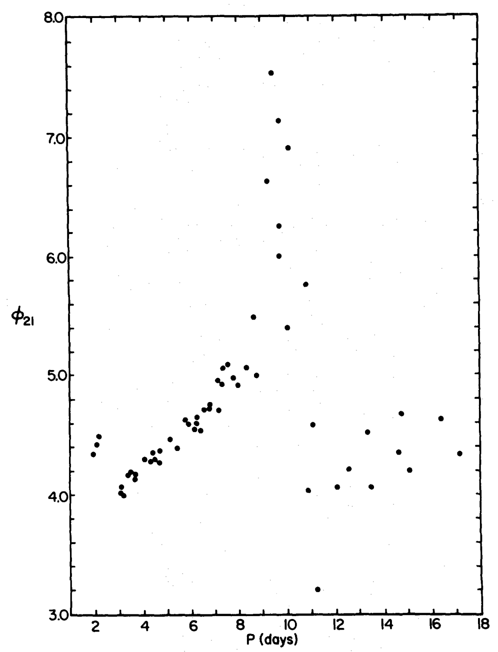

FIG. 1.—The phase difference $\phi_{21} = \phi_2 - 2\phi_1$ versus period

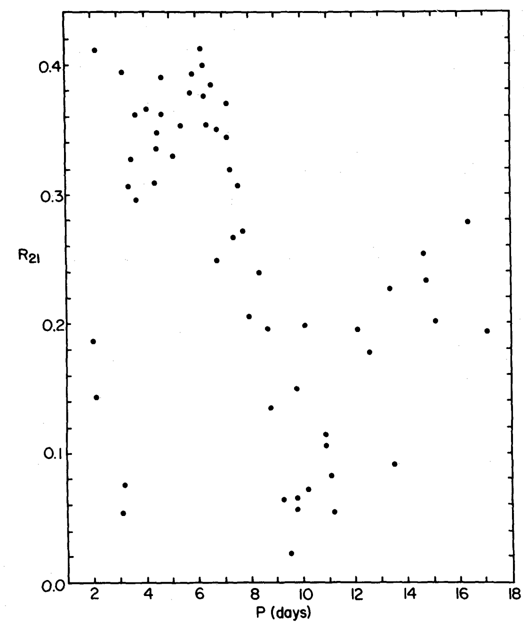

FIG. 2.—The amplitude ratio $R_{21} = A_2/A_1$ versus period

tude (Fig. 3), it is difficult to discern any hint whatsoever that these two quantities are correlated. This circumstance is repeated in the plot of ϕ_{21} versus amplitude which we have not displayed. It appears that the relationships among the lowest two Fourier coefficients are essentially governed by the resonance, with amplitude playing only a minor role.

In Figure 4 we show a plot of the phase difference ϕ_{31} versus period. Here, again, a relationship is immediately apparent, with a slow increase of ϕ_{31} at short periods, a rapid jump at about 9–10 days, followed, finally, by a milder rise with more dispersion for the longer periods. The points departing most drastically from this relationship are indicated in Figure 4 by ×. The corresponding stars are represented by the same symbol in Figure 5 which plots the amplitude ratio R_{31} against period. One notices that all of these points have small values of R_{31}, which indicates that the third Fourier term is relatively unimportant. This in turn suggests that observations more detailed than usual will be necessary to pin down ϕ_{31}. Thus the divergence of these stars from the trend in Figure 4 could be due to the possibility that ϕ_{31} is poorly determined.

Finally, returning to Figure 5, one again sees at 9–10 days a discontinuity in the amplitude ratio, although the effect is less well defined than in the case of R_{21} (Fig. 2).

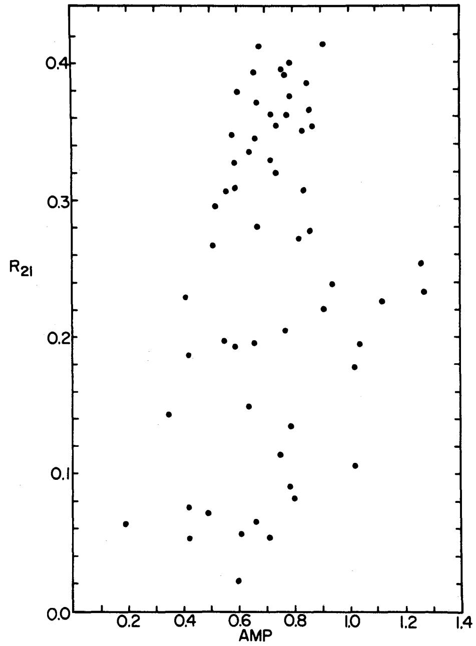

FIG. 3

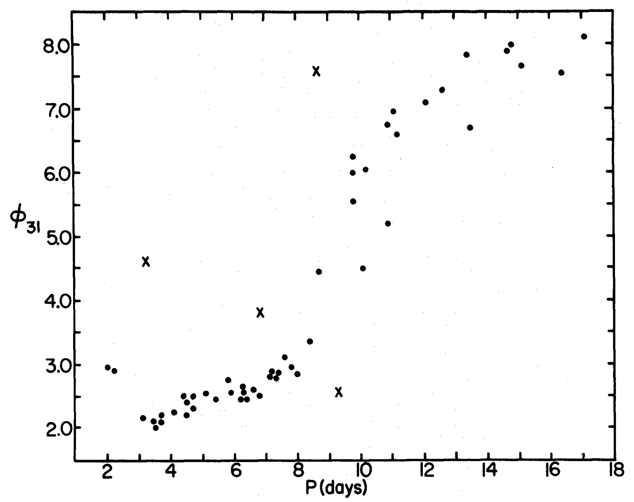

FIG. 4

FIG. 3.—R_{21} versus amplitude

FIG. 4.—The phase difference $\phi_{31} = \phi_3 - 3\phi_1$ versus period; ×, stars falling far from locus

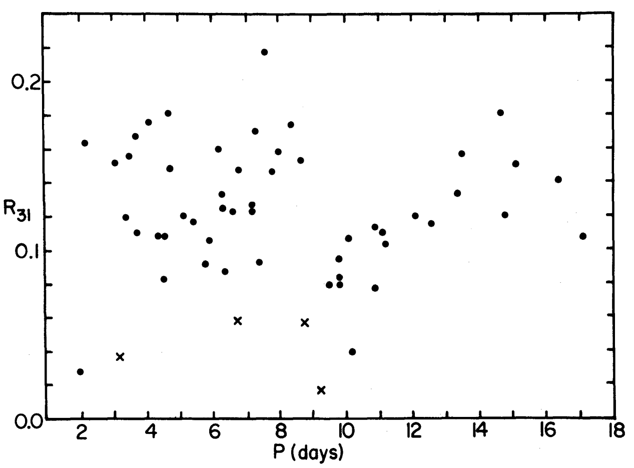

FIG. 5

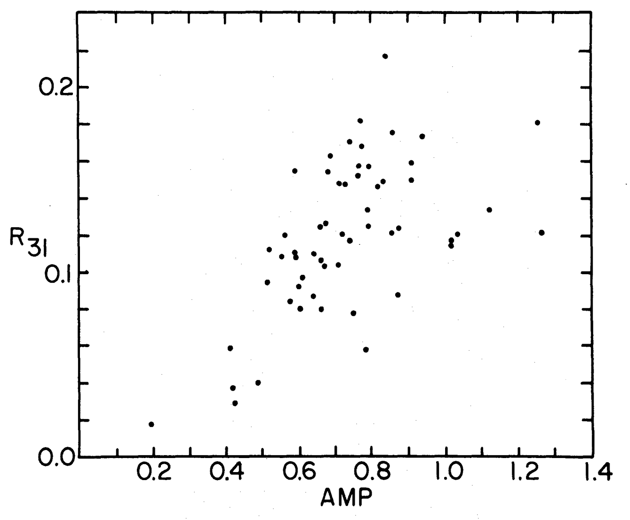

FIG. 6

FIG. 5.—The amplitude ratio $R_{31} = A_3/A_1$ versus period; ×, same stars as in Fig. 4

FIG. 6.—R_{31} versus amplitude

For comparison, Figure 6 shows R_{31} against amplitude. Here, a modest but distinct trend may be discerned with the expected positive slope. Thus, both period (i.e., the resonance) and amplitude seem to exert some effect on the third-order ratio R_{31}. On the other hand, we note that the third-order phase difference ϕ_{31} shows no apparent correlation with amplitude. Again, the plot has not been displayed.

III. DISCUSSION

In § II we have presented a quantitative description of the structure of Cepheid light curves in terms of low-order Fourier coefficients, in particular the phase differences ϕ_{21} and ϕ_{31} (Figs. 1 and 4). The relation of the coefficients to curve shape has been discussed by Simon (1977). The regularity displayed by the quantities

ϕ_{21} and ϕ_{31} (and, to a lesser extent, R_{21}) allows the possibility of direct comparison with hydrodynamic models. This can be done in a straight-forward manner by Fourier-analyzing a series of theoretical light curves and plotting for them the equivalents of Figures 1, 2, and 4.

Using this technique, quantitative evidence may be brought to bear on a number of controversial questions relating to interpretation of the Hertzsprung progression. One of these questions concerns the claim of Vermury and Stothers (1978) that the bump sequence may be reproduced with evolutionary-mass models through use of the Carson opacities. This assertion has been criticized by Hodson and Cox (1980) who propose that the "Carson bumps" are surface disturbances, not to be compared to the true features of the Hertzsprung progression. A quantitative test performed by means of Fourier decomposition could go a long way toward settling this disagreement. Similar tests might also be useful in determining whether objective differences exist between bump sequences produced by low-mass models and those emerging from the helium enrichment proposed by Cox, Michaud, and Hodson (1978).

Another outstanding problem relates to the suggestion of Davis (1979) that ascending branch features on the light curves of longer period Cepheids are due to an "artificial viscosity dip" and thus are presumably unrelated to the Hertzsprung bumps seen between 6 and 10 days. While the scatter observed at longer periods in Figures 1, 2, and 4 could be considered to support this suggestion, other possible causes for the decline in regularity have been suggested in § II. Once again, in this case, Fourier analysis of a series of theoretical models ought to provide very useful evidence.

We close this paper with a few remarks concerning the three Cepheids in our sample with periods around 2 days: SU Cas, EU Tau, and UY Eri. These stars stand out clearly in Figure 1 with values of ϕ_{21} lying considerably above the short-period locus. In Figure 4, this circumstance is repeated with ϕ_{31} for two of the objects, EU Tau having been excluded because its final (second-order) fit did not determine ϕ_{31}.

The peculiar location of these stars in the phase-period diagrams raises the question of their possibly distinct nature. The star SU Cas is a fairly well-studied Cepheid, suspected by a number of authors to pulsate in the first overtone. A recent appraisal of this object has been given by Niva and Schmidt (1979). The star EU Tau has a checkered history. Different early observers classified it both as nonvariable and as a W Ursae Majoris star. However, on the basis of an extensive series of observations, it was more recently concluded by Guinan (1972) that EU Tau is a classical Cepheid. The variable UY Eri is described by Pel (1976) as a Population II Cepheid and is indeed included by King, Cox, and Hodson (1981) on their list of BL Herculis stars. So far as we know, it is the only Population II object among the 57 stars in our sample.

Thus the three anomalous short-period variables seem on the surface to have little in common. Two speculative hypotheses which might serve to link them are the following: (1) all three are first overtone pulsators; or (2) all three belong to a population that is distinct from that of the ordinary classical Cepheids. Clearly, it is going to be necessary to secure a considerably larger sample of these objects before a study of their light curves can succeed in discerning their nature.

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